

WELL-POSEDNESS FOR A FAMILY OF PERTURBATIONS OF THE KdV EQUATION IN PERIODIC SOBOLEV SPACES OF NEGATIVE ORDER

Xavier Carvajal Paredes*
Ricardo A. Pastrán[†]

March 25, 2013

Abstract

We establish local well-posedness in Sobolev spaces $H^s(\mathbb{T})$, with $s \geq -1/2$, for the initial value problem issues of the equation

$$u_t + u_{xxx} + \eta Lu + uu_x = 0; \quad x \in \mathbb{T}, \quad t \geq 0,$$

where $\eta > 0$, $(Lu)^\wedge(k) = -\Phi(k)\widehat{u}(k)$, $k \in \mathbb{Z}$ and $\Phi \in \mathbb{R}$ is bounded above. Particular cases of this problem are the Korteweg-de Vries-Burgers equation for $\Phi(k) = -k^2$, the derivative Korteweg-de Vries-Kuramoto-Sivashinsky equation for $\Phi(k) = k^2 - k^4$, and the Ostrovsky-Stepanyams-Tsimring equation for $\Phi(k) = |k| - |k|^3$.

Keywords: Cauchy Problem, Local Well-Posedness, KdV equation.

1 Introduction

We consider the λ -periodic Cauchy problem for

$$\begin{cases} u_t + u_{xxx} + \eta Lu + uu_x = 0, & x \in [0, \lambda], \quad t \in [0, +\infty), \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

where $\eta > 0$ is a constant, the linear operator L is defined via the Fourier transform by

$$(Lu)^\wedge(k) = -\Phi(k)\widehat{u}(k), \quad \text{where } k \in \mathbb{Z}/\lambda, \quad (1.2)$$

and the Fourier symbol $\Phi(k)$ is a real valued function which is bounded above; i.e., there is a constant α such that $\Phi(k) \leq \alpha$. We take $\alpha \geq 1$ without loss of generality.

Before stating the main result of this work we give some important examples that belong to the model considered in (1.1), where $u = u(x, t)$ is a real-valued function and $\eta > 0$ is a constant. The first example is the Korteweg-de Vries-Burgers equation

$$\begin{cases} u_t + u_{xxx} - \eta u_{xx} + uu_x = 0, & t \geq 0, \\ u(x, 0) = u_0(x). \end{cases} \quad (1.3)$$

*Xavier Carvajal. Instituto de Matemáticas, Universidade Federal do Rio de Janeiro, Rio de Janeiro. E-mail: carvajal@im.ufrj.br

[†]Ricardo Pastrán. Instituto de Matemáticas, Universidade Federal do Rio de Janeiro, Rio de Janeiro. On leave from Universidad Nacional de Colombia, Bogotá. E-mail: rapastranr@unal.edu.co

Molinet and Ribaud considered the initial value problem (1.3) in [17] and proved that it is globally well-posed for given data in $H^s(\mathbb{R})$, $s > -1$, and ill-posed in $H^s(\mathbb{R})$ for $s < -1$ in the sense that one cannot solve the Cauchy problem for (1.3) by a Picard iterative method implemented on the integral formulation. They show that these results are also valid in the periodic setting. These results are surprising because the index $s = -1$ is lower than the exponents $s = -3/4$ and $s = -1/2$ which are boundaries indexes that determine the Sobolev spaces where it is possible to obtain well-posedness results using a Picard iterative method implemented on the integral formulation for the KdV equation on \mathbb{R} and \mathbb{T} , respectively. This was the first almost sharp result to a dispersive-dissipative equation using the Fourier restriction norm method or Bourgain method. It is not known what happen when $s = -1$.

Other model that fits in the family (1.1) is the derivative Korteweg-de Vries-Kuramoto Sivashinsky equation

$$\begin{cases} u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + uu_x = 0, & t \geq 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.4)$$

This equation arises as a model for long waves in a viscous fluid flowing down an inclined plane and also describes drift waves in a plasma (cf. [9, 21]). The equation (1.4) is a particular case of Benney-Lin equation [2, 21]; i.e.,

$$\begin{cases} u_t + u_{xxx} + \eta(u_{xx} + u_{xxxx}) + \beta u_{xxxxx} + uu_x = 0, & x \in \mathbb{R}, \quad t \geq 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.5)$$

when $\beta = 0$. The initial value problem associated to (1.4) was studied by Biagioni, Bona, Iorio and Scialom in [3]. They also determined the limiting behavior of solutions as the dissipation tends to zero. Biagioni and Linares proved global well-posedness for the initial value problem (1.5) for initial data in $L^2(\mathbb{R})$ in [4]. The Benney-Lin equation was studied by Chen and Li in [8] using the Fourier restriction norm method too. They proved that (1.5) is globally well posed in the Sobolev spaces $H^s(\mathbb{R})$ for $0 \geq s > -2$ and ill-posed in $H^s(\mathbb{R})$ for $s < -2$ in the sense that one cannot solve the Cauchy problem for (1.5) by a Picard iterative method implemented on the integral formulation.

Another example of this type is the Ostrovsky-Stepanyams-Tsimring (OST) equation:

$$\begin{cases} u_t + u_{xxx} - \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) + u^p u_x = 0, & t \geq 0, \quad p = 1 \\ u(x, 0) = u_0(x), \end{cases} \quad (1.6)$$

where \mathcal{H} denotes the Hilbert transform:

$$\mathcal{H}f(x) = -\frac{1}{\pi} v.p. \frac{1}{x} * f = \frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|x| > \epsilon} \frac{f(y)}{x - y} dy \quad (1.7)$$

The equation (1.6), with $p = 1$, was derived by Ostrovsky-et al. in [18] to describe the radiational instability of long waves in a stratified shear flow. The earlier well-posedness results for (1.6), with $p = 1$, can be found in [1], for given data in $H^s(\mathbb{R})$, local result when $s > 1/2$ and global result for $s \geq 1$. Carvajal and Scialom in [7] considered the initial value problem (1.6) in the real case and proved the local well-posedness results for given data in $H^s(\mathbb{R})$, $s \geq 0$ when $p = 1, 2, 3$. They also obtained the global well-posedness results for data in $L^2(\mathbb{R})$ with $p = 1$. In [11, 12] Cui and Zhao obtained a low regularity result on the (1.6) with $p = 1$ by Fourier restriction norm method. Indeed, they proved that the initial value problem (1.6) is locally well-posed in $H^s(\mathbb{R})$ for $s > -1$. Finnally, Zhao in [22] proved that (1.6) is locally well-posed in $H^s(\mathbb{R})$ for $s > -5/4$.

The next Cauchy problem of a dissipative version of the KdV equation with rough initial data

$$\begin{cases} u_t + u_{xxx} + Lu + uu_x = 0, & t \geq 0, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.8)$$

where $L = |\partial_x|^{2\gamma}$ is defined by a multiplier with symbol $|k|^{2\gamma}$ and $\gamma \geq 1$, is other example that belongs to the class (1.1). (1.8) was studied by Han and Peng in [14]. They proved working in Bourgain type space the local and global well posedness results for Sobolev spaces $H^s(\mathbb{R})$ of negative order, and the order number is lower than the well known value $-\frac{3}{4}$, i.e., $s > -s_\gamma$, where s_γ denotes the boundary index and it is given by:

$$\begin{cases} \frac{3-\gamma}{4-2\gamma}, & \text{if } 1 \leq \gamma \leq \frac{3}{2}, \\ \gamma, & \text{if } \gamma > \frac{3}{2}. \end{cases}$$

When $\gamma = 1$, this result agrees with that in [17], and it improves the result obtained in [16] in the case $\gamma \geq 1$.

Carvajal and Panthee proved in [6] local well-posedness in Sobolev spaces $H^s(\mathbb{R})$ with $s > -3/4$ to the initial value problem (1.1) but only to the real case. In particular, they obtained that result when the symbol Φ is given by

$$\Phi(\xi) = \sum_{j=0}^n \sum_{l=0}^{2m} C_{l,j} \xi^l |\xi|^j; \quad C_{l,j} \in \mathbb{R}, \quad C_{2m,n} = -1.$$

The examples above correspond to this case. They followed the theory developed by Bourgain [5] and Kenig, Ponce and Vega [15]. They used the usual Bourgain's space associated to the KdV equation instead of the Bourgain's space associated to the linear part of the initial value problem (1.28).

1.1 Notation and Main Result

We recall the theory developed by T. Tao in [20]. We define the Fourier transform of a function f defined on $[0, \lambda]$ by

$$\widehat{f}(k) = \int_0^\lambda e^{-2\pi i k x} f(x) dx \quad (1.9)$$

and we have the Fourier inversion formula

$$f(x) = \int e^{2\pi i k x} \widehat{f}(k) (dk)_\lambda \quad (1.10)$$

where $(dk)_\lambda$ is the normalized counting measure on \mathbb{Z}/λ given by

$$\int a(k) (dk)_\lambda = \frac{1}{\lambda} \sum_{k \in \mathbb{Z}/\lambda} a(k). \quad (1.11)$$

The usual properties of the Fourier transform hold:

$$\|f\|_{L^2([0,\lambda])} = \|\widehat{f}\|_{L^2((dk)_\lambda)} \quad (\text{Plancherel}), \quad (1.12)$$

$$\int_0^\lambda f(x) g(\bar{x}) dx = \int \widehat{f}(k) \widehat{\bar{g}}(k) (dk)_\lambda \quad (\text{Parseval}), \quad (1.13)$$

$$\widehat{f\bar{g}}(k) = \widehat{f} *_{\lambda} \widehat{\bar{g}}(k) = \int \widehat{f}(k - k_1) g(k_1) (dk_1)_\lambda \quad (\text{Convolution}), \quad (1.14)$$

and so on. If we apply ∂_x^m , $m \in \mathbb{N}$, to (1.10), we obtain

$$\partial_x^m f(x) = \int e^{2\pi i k x} (2\pi i k)^m \widehat{f}(k) (dk)_\lambda. \quad (1.15)$$

This, together with (1.12), motivates us to define the Sobolev space $H^s([0, \lambda])$ with the norm

$$\|f\|_{H^s([0, \lambda])} = \left\| \langle k \rangle^s \widehat{f}(k) \right\|_{L^2((dk)_\lambda)}. \quad (1.16)$$

We will often denote this space by H_λ^s for simplicity. For a function $v = v(x, t)$ which is λ -periodic with respect to the x variable and with the time variable $t \in \mathbb{R}$, we define the space-time Fourier transform $\widehat{v} = \widehat{v}(k, \tau)$ for $k \in \mathbb{Z}/\lambda$ and $\tau \in \mathbb{R}$ by

$$\widehat{v}(k, \tau) = \int_{\mathbb{R}} \int_0^\lambda e^{-2\pi i k x} e^{-2\pi i \tau t} v(x, t) dx dt. \quad (1.17)$$

This transform is inverted by

$$v(x, t) = \int_{\mathbb{R}} \int e^{2\pi i k x} e^{2\pi i \tau t} \widehat{v}(k, \tau) (dk)_\lambda d\tau. \quad (1.18)$$

Similarly, $\widehat{v}(k, t)$ and $\widehat{v}(x, \tau)$ will denote the Fourier transform of $v(x, t)$ respect to the variables x and t , respectively. C will be denote a positive constant which may be different even in a single chain of inequalities. If X, Y are Banach spaces, $\mathcal{B}(X; Y)$ is the space of the linear continue operators of X in Y with the norm $\|T\|_{X \rightarrow Y} = \sup_{\|x\|_X=1} \|Tx\|_Y$. If $X = Y$ we will write $\mathcal{B}(X)$ inside of $\mathcal{B}(X; X)$. The solution to the linear KdV equation:

$$\begin{cases} u_t + u_{xxx} = 0, & x \in [0, \lambda], \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.19)$$

is given by

$$u(x, t) = U_\lambda(t) u_0(x) = \int e^{2\pi i k x} e^{-(2\pi i k)^3 t} \widehat{u_0}(k) (dk)_\lambda, \quad (1.20)$$

which may be rewritten as a space-time inverse Fourier transform,

$$U_\lambda(t) u_0(x) = \int_{\mathbb{R}} \int e^{2\pi i \tau t} e^{2\pi i k x} \delta(\tau - 4\pi^2 k^3) \widehat{u_0}(k) (dk)_\lambda d\tau, \quad (1.21)$$

where $\delta(\kappa)$ represents a 1-dimensional Dirac mass at $\kappa = 0$. This shows that $U_\lambda(\cdot) u_0$ has its space-time Fourier transform supported precisely on the cubic $\tau = 4\pi^2 k^3$ in $\mathbb{Z}/\lambda \times \mathbb{R}$. So, we recall the known Bourgain's space associated to the KdV equation. For $s, b \in \mathbb{R}$, we define the $\mathcal{Y}_{s,b}([0, \lambda] \times \mathbb{R})$ spaces for λ -periodic KdV via the norm

$$\begin{aligned} \|u\|_{\mathcal{Y}_{s,b}([0, \lambda] \times \mathbb{R})} &\equiv \left\| \langle \tau - 4\pi^2 k^3 \rangle^b \langle k \rangle^s \widehat{u}(k, \tau) \right\|_{L^2((dk)_\lambda) L_\tau^2} = \left\| \langle \tau \rangle^b \langle k \rangle^s (U_\lambda(-t)u)^\wedge(k, \tau) \right\|_{L^2((dk)_\lambda) L_\tau^2} \\ &= \left(\int_{\mathbb{R}} \int \langle \tau \rangle^{2b} \langle k \rangle^{2s} |(U_\lambda(-t)u)^\wedge(k, \tau)|^2 d\tau (dk)_\lambda \right)^{1/2}. \end{aligned} \quad (1.22)$$

Remark 1.1. The spatial mean $\int_{\mathbb{T}} u(x, t) dx$ is conserved during the evolution of the KdV equation. We may assume that the initial data ϕ satisfies a mean-zero assumption $\int_{\mathbb{T}} \phi(x) dx$ since otherwise we can replace the dependent variable u by $v = u - \int_{\mathbb{T}} \phi$ at the expense of a harmless linear first order term. This observation was used by Bourgain in [5].

Since the λ -periodic initial value problem for KdV is equivalent to the integral equation

$$u(t) = U_\lambda(t) \phi - \frac{1}{2} \int_0^t U_\lambda(t - t') \partial_x (u^2(t')) dt', \quad (1.23)$$

the study of periodic KdV in [5] and [15] was based in solve (1.23) using the contraction principle in the Bourgain's spaces $\mathcal{Y}_{s,1/2}$ which was possible in virtue of the optimal bilinear estimate for $\partial_x u^2$, from Kenig, Ponce and Vega in the periodic case:

Proposition 1.1 ([15]). *For $s \in (-1/2, 0]$ it follows that*

$$\left\| \frac{1}{2} \partial_x u^2 \right\|_{\mathcal{Y}_{s, -1/2}} \leq c \|u\|_{\mathcal{Y}_{s, 1/2}}^2.$$

For the case $s = -1/2$, see the Corollary 6.5 in [20]. So, it was proved that the initial value problem for KdV on \mathbb{T} is locally well-posed for $s \geq -1/2$. The space $\mathcal{Y}_{s, 1/2}$ barely fails to control the $L_t^\infty H_x^s$ norm. To ensure continuity of the time flow of the solution Colliander, Keel, Staffilani, Takaoka and Tao, in [10], introduced the slightly smaller space Y^s defined via the norm

$$\|u\|_{Y^s} = \|u\|_{\mathcal{Y}_{s, 1/2}} + \|\langle k \rangle^s \widehat{u}(k, \tau)\|_{L^2((dk)_\lambda) L^1(d\tau)}, \quad (1.24)$$

and the companion space Z^s defined via the norm

$$\|u\|_{Z^s} = \|u\|_{\mathcal{Y}_{s, -1/2}} + \left\| \frac{\langle k \rangle^s \widehat{u}(k, \tau)}{\langle \tau - 4\pi^2 k^3 \rangle} \right\|_{L^2((dk)_\lambda) L^1(d\tau)}. \quad (1.25)$$

Note that, if $u \in Y^s$, then $u \in L_t^\infty H_x^s$. Thus, they solve the integral equation (1.23) based around iteration in the space Y^s . They obtained the bilinear estimate for $\partial_x u^2$:

Proposition 1.2. *If u and v are λ -periodic functions of x , also depending upon t having zero x -mean for all t , then*

$$\|\Psi(t) \partial_x(uv)\|_{Z^{-1/2}} \lesssim \lambda^{0+} \|u\|_{\mathcal{Y}_{-1/2, 1/2}} \|v\|_{\mathcal{Y}_{-1/2, 1/2}}. \quad (1.26)$$

where $\Psi \in C_0^\infty(\mathbb{R})$ is a cut-off function such that $0 \leq \Psi(t) \leq 1$ and is supported on $[-2, 2]$ with $\Psi = 1$ on $[-1, 1]$.

Remark 1.2. *Note that (1.26) implies $\|\Psi(t) \partial_x(uv)\|_{Z^{-1/2}} \lesssim \lambda^{0+} \|u\|_{Y^{-1/2}} \|v\|_{Y^{-1/2}}$.*

So, Colliander, Keel, Staffilani, Takaoka and Tao in [10] reproved that the initial value problem for KdV equation on \mathbb{T} is locally well-posed for $s \geq -1/2$. Our interest here is to obtain well-posedness results for the λ -periodic initial value problem (1.1) with given data u_0 in the Sobolev space H_λ^s of negative order:

Theorem 1.1 (Main Result). *The initial value problem (1.1) with $\eta > 0$ and L given by (1.2) is locally well-posed for any data $u_0 \in H^s(\mathbb{T})$, for $s \geq -1/2$.*

To prove this theorem we use Bourgain's type space. So, we should be able to write (1.1) for all $t \in \mathbb{R}$. For this, we define

$$\eta(t) \equiv \eta \operatorname{sgn}(t) = \begin{cases} \eta, & \text{if } t \geq 0, \\ -\eta, & \text{if } t < 0, \end{cases} \quad (1.27)$$

and write (1.1) in the form

$$\begin{cases} u_t + u_{xxx} + \eta(t)Lu + uu_x = 0, & x \in [0, \lambda], \quad t \in \mathbb{R}, \\ u(x, 0) = u_0(x). \end{cases} \quad (1.28)$$

We first want to build a representation formula for the solution of the linearization of (1.1) about the zero solution. So, we wish to solve the linear homogeneous λ -periodic initial value problem

$$\begin{cases} w_t + w_{xxx} + \eta(t)Lw = 0, & x \in [0, \lambda], \quad t \in \mathbb{R}, \\ w(x, 0) = w_0(x). \end{cases} \quad (1.29)$$

The Fourier inversion formula (1.10) allows us to write the solution of (1.29):

$$w(x, t) = V_\lambda(t) w_0(x) = \int e^{2\pi i k x} e^{-(2\pi i k)^3 t + \eta \Phi(k)|t|} \widehat{w_0}(k) (dk)_\lambda. \quad (1.30)$$

Observe that, defining $\widetilde{U}_\lambda(t)$ by

$$(\widetilde{U}_\lambda(t)w_0)^\wedge(k) = e^{\eta|t|\Phi(k)}\widehat{w_0}(k), \quad k \in \mathbb{Z}/\lambda,$$

the semigroup $V_\lambda(t)$ can be written as $V_\lambda(t) = U_\lambda(t)\widetilde{U}_\lambda(t)$ where $U_\lambda(t)$ is the unitary group of the KdV (1.21). We next find a representation for the solution of the linear inhomogeneous λ -periodic initial value problem

$$\begin{cases} v_t + v_{xxx} + \eta(t)Lv = F, & x \in [0, \lambda], \quad t \in \mathbb{R}, \\ v(x, 0) = 0, \end{cases} \quad (1.31)$$

with $F = F(x, t)$ a given time-dependent λ -periodic (in x) function. By Duhamel's principle,

$$v(x, t) = \int_0^t V_\lambda(t-t') F(x, t') dt'. \quad (1.32)$$

We apply (1.30), rewrite $\widehat{F}(k, t')$ using the Fourier inversion formula in the time variable and rearrange integrations to find

$$v(x, t) = \int_{\mathbb{R}} \int e^{2\pi i k x} e^{2\pi i (4\pi^2 k^3)t + \eta\Phi(k)|t|} \int_0^t e^{[2\pi i (\tau - 4\pi^2 k^3) - \eta(t)\Phi(k)]t'} dt' \widehat{F}(k, \tau) (dk)_\lambda d\tau. \quad (1.33)$$

Performing the t' -integration, we find

$$v(x, t) = \int_{\mathbb{R}} \int e^{2\pi i k x} e^{2\pi i (4\pi^2 k^3)t + \eta\Phi(k)|t|} \frac{e^{[2\pi i (\tau - 4\pi^2 k^3) - \eta(t)\Phi(k)]t} - 1}{2\pi i (\tau - 4\pi^2 k^3) - \eta(t)\Phi(k)} \widehat{F}(k, \tau) (dk)_\lambda d\tau. \quad (1.34)$$

Then, the λ -periodic initial value problem for (1.28) is equivalent to the integral equation

$$u(t) = V_\lambda(t)u_0 - \frac{1}{2} \int_0^t V_\lambda(t-t') \partial_x(u^2(t')) dt'. \quad (1.35)$$

The integral equation (1.35) can be solved using the contraction principle in the space $\mathcal{Y}_{s,1/2}$ following the ideas of Carvajal and Panthee in [6]. The main difficulty to resolve it of this way is the periodic bilinear estimate for $\partial_x u^2$, given in the Proposition 1.1, because it's very restrictive compared with the bilinear estimate (see Theorem 1.1 in [15]) of the real case in which $b \in (1/2, 1)$. So, we shall obtain a refined estimative to the forcing term of the integral equation associated to (1.28) which will permits us to use the Proposition 1.1 to solve (1.35). This refinement is made in the Proposition 3.1 but we prove our main result via the contraction principle in the space Y^s and the bilinear estimate (1.26) from Colliander, Keel, Staffilani, Takaoka and Tao.

The layout of this paper is as follows. In Section 2 we present some basic results. In Section 3 we give the boundedness results for linear operators involving the spaces Y^s , Z^s and $\mathcal{Y}_{s,1/2}$. The proof of the main Theorem 1.1 will be given in Section 4.

2 Preliminary Results

Lemma 2.1. *Let $a \leq 0$, $\psi \in C_0^\infty$ with support in $[-2, 2]$ and $\psi_T(t) = \psi(t/T)$. Then,*

$$\left\| \psi_T(t) \int_0^t e^{a|t-x|} g(x) dx \right\|_{L^2} \leq \frac{C(1+T)}{1+|a|} \|g\|_{L^2}. \quad (2.1)$$

If $g(0) = 0$,

$$\left\| \psi_T(t) \frac{d}{dt} \int_0^t e^{a|t-x|} g(x) dx \right\|_{L^2} \leq \frac{C(1+T)}{1+|a|} \left\| \frac{dg}{dt} \right\|_{L^2}, \quad (2.2)$$

and,

$$\|sgn(\cdot)g(\cdot)\|_{H^1} \leq \|g(\cdot)\|_{H^1}. \quad (2.3)$$

$C = C_\psi = \max\left\{\|\psi\|_{L^\infty}, \left\|\frac{d\psi}{dt}\right\|_{L^\infty}\right\}$ is a constant depending on ψ .

Proof. We are going to argue by duality to obtain (2.1). We take $\varphi \in L^2$ with $\|\varphi\|_{L^2} \leq 1$. Then,

$$\begin{aligned} \int_{\mathbb{R}} \varphi(t) \left\{ \psi_T(t) \int_0^t e^{a|t-x|} g(x) dx \right\} dt &= \int_{-2T}^{2T} \int_0^t \varphi(t) \left\{ \psi_T(t) e^{a|x|} g(t-x) \right\} dx dt \\ &= \int_{-2T}^0 e^{a|x|} \int_{-2T}^x \varphi(t) \psi_T(t) g(t-x) dt dx + \int_0^{2T} e^{a|x|} \int_x^{2T} \varphi(t) \psi_T(t) g(t-x) dt dx \\ &\leq 2 \int_{-2T}^{2T} e^{a|x|} \|\varphi\|_{L^2} \|\psi_T(\cdot)g(\cdot-x)\|_{L^2} dx \leq \frac{C}{|a|} \|g\|_{L^2}. \end{aligned} \quad (2.4)$$

It was used the Cauchy-Schwartz's inequality to obtain the first inequality in (2.4). Also, it is truth that

$$\int_{-2T}^{2T} e^{a|x|} \|\varphi\|_{L^2} \|\psi_T(\cdot)g(\cdot-x)\|_{L^2} dx \leq C T \|g\|_{L^2}. \quad (2.5)$$

From (2.4) and (2.5), we conclude (2.1). We use that $g(0) = 0$ to obtain

$$\frac{d}{dt} \left(\int_0^t e^{a|t-x|} g(x) dx \right) = \int_0^t e^{a|x|} \frac{dg}{dt}(t-x) dx = \int_0^t e^{a|t-x|} \frac{dg}{dx}(x) dx$$

and this, together with (2.1), implies (2.2). An easy computation shows (2.3). \square

Remark 2.1. We consider a cut-off function $\Psi \in C^\infty(\mathbb{R})$, such that $0 \leq \Psi(t) \leq 1$,

$$\Psi(t) = \begin{cases} 1, & \text{if } |t| \leq 1 \\ 0, & \text{if } |t| \geq 2. \end{cases}$$

Let us define $\Psi_T(t) = \Psi(\frac{t}{T})$ and $\widetilde{\Psi}_T(t) = sgn(t)\Psi_T(t)$. Note that multiplication by $\Psi(t)$ is a bounded operation on the spaces Y^s , Z^s and $\mathcal{Y}_{s,b}$.

The next result will allow us to prove the Lemma 3.5 and to reduce the proof of (3.55).

Proposition 2.1. Let $0 \leq b \leq 1$, α_1, α_2 negatives and $a = \alpha_1 + \alpha_2$. Then,

$$\left\| \Psi_T(t) \int_0^t e^{a|t-x|} f(x) dx \right\|_{H^b} \leq C(1+T) \left\| \Psi_{2T}(t) \int_0^t e^{\alpha_2|t-x|} f(x) dx \right\|_{H^b}, \quad (2.6)$$

where $C = C_\Psi = \max\left\{\|\Psi\|_{L^\infty}, \left\|\frac{d\Psi}{dt}\right\|_{L^\infty}\right\}$ is a constant depending on Ψ .

Proof. Let $g(t) = \int_0^t e^{\alpha_2|t-x|} f(x) dx$. Thus, $\frac{dg}{dt}(t) = f(t) + \alpha_2 sgn(t)g(t)$. Integrating by parts, we have

$$\begin{aligned} \int_0^t e^{a|t-x|} f(x) dx &= \int_0^t e^{a(|t|-|x|)} \frac{dg}{dx}(x) dx - \alpha_2 \int_0^t e^{a|t-x|} sgn(x) g(x) dx \\ &= g(t) + \alpha_1 sgn(t) \int_0^t e^{a|t-x|} g(x) dx. \end{aligned} \quad (2.7)$$

We obtain (2.6) when $b = 0$ as consequence of (2.7), (2.1) and

$$\left\| \alpha_1 \operatorname{sgn}(t) \Psi_T(t) \int_0^t e^{a|t-x|} g(x) dx \right\|_{L^2} \leq \frac{|\alpha_1| C (1+T)}{1 + |\alpha_1| + |\alpha_2|} \|\Psi_{2T} g\|_{L^2}.$$

Now, we are going to obtain (2.6) when $b = 1$. We know from (2.7) that

$$\left\| \Psi_T(t) \int_0^t e^{a|t-x|} f(x) dx \right\|_{H^1} \leq \|\Psi_T(t) g(t)\|_{H^1} + |\alpha_1| \left\| \operatorname{sgn}(t) \Psi_T(t) \int_0^t e^{a|t-x|} g(x) dx \right\|_{H^1}. \quad (2.8)$$

Since $\|\Psi_T g\|_{H^1} \leq C \|\Psi_{2T} g\|_{H^1}$, by virtue of (2.3) it is sufficient to estimate

$$\left\| \frac{d}{dt} \left(\Psi_T(t) \int_0^t e^{a|t-x|} g(x) dx \right) \right\|_{L^2}$$

which is bounded by

$$\left\| \frac{d\Psi_T}{dt}(t) \left(\int_0^t e^{a|t-x|} g(x) dx \right) \right\|_{L^2} + \left\| \Psi_T(t) \frac{d}{dt} \left(\int_0^t e^{a|t-x|} g(x) dx \right) \right\|_{L^2}. \quad (2.9)$$

For the first term above we can apply (2.1) and

$$\left\| \frac{d\Psi}{dt}(t/T) \int_0^T e^{a|t-x|} (\Psi_{2T} g)(x) dx \right\|_{L^2} \leq \frac{C(T+1)}{1+|a|} \|\Psi_{2T} g\|_{L^2} \leq \frac{C(T+1)T}{1+|a|} \left\| \frac{d}{dt} (\Psi_{2T} g) \right\|_{L^2},$$

where, in the last inequality, it was used that

$$\|\Psi_{2T} g\|_{L^2}^2 = \int_{-4T}^{4T} |\Psi_{2T}(t) g(t)|^2 dt \leq C T \|\Psi_{2T} g\|_{L^\infty}^2 \leq C T \|\Psi_{2T} g\|_{L^2} \left\| \frac{d}{dt} (\Psi_{2T} g) \right\|_{L^2}.$$

For the second term from (2.9) we used (2.2) with $\Psi_{2T} g$ instead g because $g = \Psi_{2T} g$ on $[-T, T]$. This implies (2.6) when $b = 1$. The result (2.6) is obtained interpolating the cases $b = 0$ and $b = 1$. \square

The following Lemma plays a central role estimating the free term of the integral equation (1.35). This Lemma allows us to work in the usual $\mathcal{Y}_{s,1/2}$ space associated to the KdV equation.

Lemma 2.2. *Let $0 < T \lesssim 1$ and $a \leq \alpha$. Then we have*

$$\|\Psi_T(\cdot)\|_{H_t^b} \leq C(T^{1/2} + T^{1/2-b}) \quad \forall b \geq 0, \quad (2.10)$$

$$\left\| \Psi_T(\cdot) e^{a|\cdot|} \right\|_{H_t^{1/2}} \leq C e^{2\alpha}, \quad (2.11)$$

$$\left\| \Psi_T(\cdot) e^{a|\cdot|} \right\|_{L_t^1} \leq C e^{2\alpha}, \quad (2.12)$$

$$|(|t| \Psi_T(t) e^{a|t|})^\wedge(\tau)| \leq \frac{C T^2}{1 + (\tau^2 + a^2) T^2}, \quad (2.13)$$

where $C = C_\psi = \max\left\{ \|\psi\|_{L^\infty}, \left\| \frac{d\psi}{dt} \right\|_{L^\infty}, \left\| \frac{d^2\psi}{dt^2} \right\|_{L^\infty} \right\}$ is a constant depending on ψ .

Proof. It's clear that

$$\|\Psi_T\|_{L^2}^2 = \int_{\mathbb{R}} \left| \Psi\left(\frac{t}{T}\right) \right|^2 dt = \int_{\mathbb{R}} T |\Psi(t)|^2 dt = T \|\Psi\|_{L^2}^2. \quad (2.14)$$

By the definition of the space H^b , we have

$$\|\Psi_T\|_{H_t^b} \leq C \|\Psi_T\|_{L^2} + C \|D_t^b \Psi_T\|_{L^2} = C T^{1/2} \|\Psi\|_{L^2} + C T^{1/2-b} \|D_t^b \Psi\|_{L^2}, \quad (2.15)$$

where we have used the fact

$$\|D_t^b \Psi_T\|_{L^2}^2 = \int_{\mathbb{R}} |\tau|^{2b} |T \widehat{\Psi}(T\tau)|^2 d\tau = T^{1-2b} \|D_t^b \Psi\|_{L^2}^2.$$

Since $\|\Psi\|_{L^2}$ and $\|D_t^b \Psi\|_{L^2}$ are bounded by a constant because of the form of the function Ψ , then from (2.15) we obtain (2.10). We call $h(t) = \Psi(t) e^{a|t|T}$, and so $h_T(t) = \Psi_T(t) e^{a|t|}$, to get like in (2.15):

$$\|\Psi_T(\cdot) e^{a|\cdot|}\|_{H_t^{1/2}} = \|h_T\|_{H_t^{1/2}} \leq C T^{1/2} \|h\|_{L^2} + C \|D_t^{1/2} h\|_{L^2}. \quad (2.16)$$

We know that

$$\|h\|_{L^2}^2 = \int_{-2T}^{2T} |\Psi(t)|^2 e^{2a|t|T} dt \leq 4T e^{4\alpha T^2} \|\Psi\|_{L^\infty}^2. \quad (2.17)$$

To bounded the term $\|D_t^{1/2} h\|_{L^2}$ we are going to explore $\widehat{h}(\tau)$ integrating by parts two times,

$$\begin{aligned} \widehat{h}(\tau) &= \int_0^{+\infty} \Psi(t) e^{aTt} e^{-it\tau} dt + \int_{-\infty}^0 \Psi(t) e^{-aTt} e^{-it\tau} dt \\ &= \frac{-1}{aT - i\tau} \left(1 + \int_0^{+\infty} \frac{d\Psi}{dt}(t) e^{t(aT - i\tau)} dt \right) - \frac{1}{aT + i\tau} \left(1 - \int_{-\infty}^0 \frac{d\Psi}{dt}(t) e^{-t(aT + i\tau)} dt \right) \\ &= \frac{-2aT}{(aT)^2 + \tau^2} + \frac{1}{(aT - i\tau)^2} \int_0^{+\infty} \frac{d^2\Psi}{dt^2}(t) e^{t(aT - i\tau)} dt + \frac{1}{(aT + i\tau)^2} \int_{-\infty}^0 \frac{d^2\Psi}{dt^2}(t) e^{-t(aT + i\tau)} dt. \end{aligned} \quad (2.18)$$

From this we have that

$$|\widehat{h}(\tau)| \leq \frac{2|a|T}{(aT)^2 + \tau^2} + \frac{2(2T) e^{2\alpha T^2} \left\| \frac{d^2\Psi}{dt^2} \right\|_{L^\infty}}{(aT)^2 + \tau^2}, \quad (2.19)$$

and, from (2.18)

$$|\widehat{h}(\tau)| \leq 4T e^{2\alpha T^2} \|\Psi\|_{L^\infty} \leq 4 e^{2\alpha} \|\Psi\|_{L^\infty} = C_1 e^{2\alpha}. \quad (2.20)$$

Hence, with $C_0 e^{2\alpha} = 4 e^{2\alpha} \left\| \frac{d^2\Psi}{dt^2} \right\|_{L^\infty} \geq 4T e^{2\alpha T^2} \left\| \frac{d^2\Psi}{dt^2} \right\|_{L^\infty}$, from (2.19) and (2.20), we obtain that

$$|\widehat{h}(\tau)| \leq \frac{2|a|T + C e^{2\alpha}}{1 + (aT)^2 + \tau^2}, \quad (2.21)$$

where $C = C_0 + C_1$. Multiplying by $|\tau|^{1/2}$ in (2.21), taking square and integrating on \mathbb{R} , we have that

$$\begin{aligned} \|D_t^{1/2} h\|_{L^2}^2 &= \left\| |\tau|^{1/2} \widehat{h}(\tau) \right\|_{L^2}^2 \lesssim 4a^2 T^2 \int_{\mathbb{R}} \frac{|\tau|}{(1 + a^2 T^2 + \tau^2)^2} d\tau + C e^{4\alpha} \int_{\mathbb{R}} \frac{|\tau|}{(1 + a^2 T^2 + \tau^2)^2} d\tau \\ &\lesssim 4a^2 T^2 \int_{\mathbb{R}} \frac{|\tau|}{(a^2 T^2 + \tau^2)^2} d\tau + C e^{4\alpha} \int_{\mathbb{R}} \frac{|\tau|}{(1 + \tau^2)^2} d\tau \\ &\lesssim 4 + C e^{4\alpha} \leq C e^{4\alpha}, \end{aligned} \quad (2.22)$$

where in the second inequality we used $\tau = |a|Tx$. From (2.16), (2.17), (2.22) and since $T \leq 1$, we conclude (2.11). Integrating on \mathbb{R} the next inequality which is consequence of (2.21)

$$|\widehat{h}(\tau)| \lesssim \frac{|a|T}{(aT)^2 + \tau^2} + \frac{C e^{2\alpha}}{1 + \tau^2}.$$

we have proved (2.12). The proof of (2.13) is equal to that of (2.6) in the Lemma 2.3 in [6]. \square

3 Linear Estimates

Here we study the linear operator ΨV_λ as well as the linear operator M_λ defined as

$$M_\lambda : f \longmapsto \Psi(t) \int_0^t V_\lambda(t-t') f(t') dt'. \quad (3.1)$$

3.1 Linear Estimates for the Free Term in Y^s

The next proposition gives a bounded to the free term of the integral equation (1.35).

Lemma 3.1.

$$\|\Psi(t) V_\lambda(t) \phi\|_{Y^s} \lesssim \|\phi\|_{H^s}. \quad (3.2)$$

Proof. We denote $\Theta_k(t) = \Psi(t) e^{\eta \Phi(k)|t|}$. Then

$$(\Psi(t) V_\lambda(t) \phi)^\wedge(k, \tau) = \widehat{\Theta_k(t)} * (e^{-(2\pi i k)^3 t})^\wedge(\tau) \widehat{\phi}(k) = \widehat{\Theta_k}(\tau - 4\pi^2 k^3) \widehat{\phi}(k). \quad (3.3)$$

So, $\|\Psi(t) V_\lambda(t) \phi\|_{Y^s}^2$

$$\begin{aligned} &\leq \left\| \langle \tau - 4\pi^2 k^3 \rangle^{1/2} \langle k \rangle^s \widehat{\Theta_k}(\tau - 4\pi^2 k^3) \widehat{\phi}(k) \right\|_{L^2((dk)_\lambda) L^2(d\tau)}^2 + \left\| \langle k \rangle^s \widehat{\Theta_k}(\tau - 4\pi^2 k^3) \widehat{\phi}(k) \right\|_{L^2((dk)_\lambda) L^1(d\tau)}^2 \\ &= \left(\left\| \langle \tau \rangle^{1/2} \widehat{\Theta_k}(\tau) \right\|_{L^2(d\tau)}^2 + \left\| \widehat{\Theta_k}(\tau) \right\|_{L^1(d\tau)}^2 \right) \left\| \langle k \rangle^s \widehat{\phi}(k) \right\|_{L^2((dk)_\lambda)}^2. \end{aligned} \quad (3.4)$$

(3.4) with (2.11) and (2.12) imply (3.2). \square

3.2 Linear Estimates for the Forcing Term in Y^s

Lemma 3.2.

$$\left\| \Psi(t) \int_0^t V_\lambda(t-t') F(t') dt' \right\|_{Y^s} \lesssim \|F\|_{Z^s}. \quad (3.5)$$

Proof. By applying a smooth cutoff function, we may assume that F is supported on $\mathbb{T} \times [-3, 3]$. Let $a(t) = \text{sgn}(t)b(t)$, where b is a smooth bump function supported on $[-10, 10]$ which equals 1 on $[-5, 5]$. The identity

$$\chi_{[0,t]}(t') = \frac{1}{2} (a(t') + a(t-t')),$$

valid for $t \in [-2, 2]$ and $t' \in [-3, 3]$, allows us to rewrite

$$\begin{aligned} \Psi(t) \int_0^t V_\lambda(t-t') F(t') dt' &= \Psi(t) \int_{\mathbb{R}} \chi_{[0,t]}(t') V_\lambda(t-t') F(t') dt' \\ &= \frac{1}{2} \Psi(t) V_\lambda(t) \int_{\mathbb{R}} a(t') V_\lambda(-t') F(t') dt' + \frac{1}{2} \Psi(t) \int_{\mathbb{R}} a(t-t') V_\lambda(t-t') F(t') dt'. \end{aligned} \quad (3.6)$$

We consider the contribution of each one of the addend of (3.6). We denote $\tilde{a}_k(t') = a(t') e^{\eta \Phi(k)|t'|}$ and we use (3.2) to obtain

$$\begin{aligned} \left\| \Psi(t) V_\lambda(t) \int_{\mathbb{R}} a(t') V_\lambda(-t') F(t') dt' \right\|_{Y^s} &\leq \left\| \int_{\mathbb{R}} a(t') V_\lambda(-t') F(t') dt' \right\|_{H^s} \\ &= \left\| \langle k \rangle^s \int_{\mathbb{R}} e^{-2\pi i (4\pi^2 k^3) t'} \tilde{a}_k(t') \widehat{F}(k, t') dt' \right\|_{L^2((dk)_\lambda)} \\ &= \left\| \langle k \rangle^s \int_{\mathbb{R}} \widehat{\tilde{a}_k(t')} (4\pi^2 k^3 - \tau) \widehat{F}(k, \tau) d\tau \right\|_{L^2((dk)_\lambda)}. \end{aligned} \quad (3.7)$$

As in the proof of (2.21), integrating by part twice we obtain

$$|\widehat{a}_k(\tau)| \leq \frac{C(\eta, \alpha)(1 + |\tau|)}{1 + \eta^2 \Phi(k)^2 + \tau^2} \leq \frac{C(\eta, \alpha)}{1 + |\tau|}.$$

We have from (3.7) that

$$\left\| \Psi(t) V_\lambda(t) \int_{\mathbb{R}} a(t') V_\lambda(-t') F(t') dt' \right\|_{Y^s} \leq C(\eta, \alpha) \left\| \int_{\mathbb{R}} \frac{\langle k \rangle^s \widehat{F}(k, \tau)}{\langle \tau - 4\pi^2 k^3 \rangle} d\tau \right\|_{L^2((dk)_\lambda)} \quad (3.8)$$

The contribution of the second term in (3.6) is calculated using that multiplication by $\Psi(t)$ is a bounded operation on the space Y^s (See Remark 2.1) and note that the space-time Fourier transform of $\int_{\mathbb{R}} a(t - t') V_\lambda(t - t') F(t') dt'$ is given by

$$\begin{aligned} \left(\int_{\mathbb{R}} a(t - t') V_\lambda(t - t') F(t') dt' \right)^\wedge(k, \tau) &= \left(\int_{\mathbb{R}} a(t - t') e^{-(2\pi i k)^3(t-t') + \eta \Phi(k) |t-t'|} \widehat{F}(k, t') dt' \right)^\wedge(\tau) \\ &= \left(\widehat{a}_k(\cdot) e^{2\pi i(4\pi^2 k^3)(\cdot)} * \widehat{F}(k, \cdot)(t) \right)^\wedge(\tau) = \widehat{a}_k(\tau - 4\pi^2 k^3) \widehat{F}(k, \tau). \end{aligned} \quad (3.9)$$

From the definitions (1.24), (1.25) and from the estimate for \widehat{a} used above we have:

$$\begin{aligned} &\left\| \int_{\mathbb{R}} a(t - t') V_\lambda(t - t') F(t') dt' \right\|_{Y^s} \\ &= \left\| \langle \tau - 4\pi^2 k^3 \rangle^{1/2} \langle k \rangle^s \widehat{a}_k(\tau - 4\pi^2 k^3) \widehat{F}(k, \tau) \right\|_{L^2((dk)_\lambda) L^2(d\tau)} + \left\| \langle k \rangle^s \widehat{a}_k(\tau - 4\pi^2 k^3) \widehat{F}(k, \tau) \right\|_{L^2((dk)_\lambda) L^1(d\tau)} \\ &\leq C(\eta, \alpha) \left\| \langle \tau - 4\pi^2 k^3 \rangle^{-1/2} \langle k \rangle^s \widehat{F}(k, \tau) \right\|_{L^2((dk)_\lambda) L^2(d\tau)} + C(\eta, \alpha) \left\| \frac{\langle k \rangle^s \widehat{F}(k, \tau)}{\langle \tau - 4\pi^2 k^3 \rangle} \right\|_{L^2((dk)_\lambda) L^1(d\tau)}. \end{aligned} \quad (3.10)$$

(3.8) and (3.10) give (3.5). \square

3.3 Linear Estimates for the Forcing Term in $\mathcal{Y}_{s,1/2}$

Proposition 3.1. *Let $T \in [\frac{\sqrt{2}}{\sqrt{\beta}}, \frac{1}{2}]$, $s \in \mathbb{R}$ and $\beta > 8$. Then,*

$$\left\| \Psi_T(t) \int_0^t V_\lambda(t - t') F(t') dt' \right\|_{\mathcal{Y}_{s,1/2}} \leq C \eta \alpha (\beta + (\eta \alpha)^2) e^{2\eta \alpha} T^{1/2} \|F\|_{\mathcal{Y}_{s,-1/2}}. \quad (3.11)$$

where C is a constant.

Remark 3.1. *The Proposition 3.1 together with the inequality (2.11), which implies a linear estimate for the free term in $\mathcal{Y}_{s,1/2}$, that is, $\|\Psi(t) V_\lambda(t) \phi\|_{\mathcal{Y}_{s,1/2}} \leq C e^{2\alpha} \|\phi\|_{H^s}$, and the bilinear estimate from Kenig, Ponce and Vega given in the Proposition 1.1 guarantee the local well-posedness result to the λ -periodic initial value problem (1.1) in the Sobolev spaces $H^s(\mathbb{T})$ to $s > -1/2$ at least for small initial data.*

To prove this Proposition 3.1 we need the next Lemmas:

Lemma 3.3 (Schur's Lemma). *Let f be in $\mathcal{S}(\mathbb{R})$ and L the integral operator, given by*

$$(Lf)(x) = \int_{\mathbb{R}} N(x, y) f(y) dy$$

where the kernel N is such that

$$\sup_x \int_{\mathbb{R}} |N(x, y)| dy \leq 1, \quad \text{and} \quad \sup_y \int_{\mathbb{R}} |N(x, y)| dx \leq 1.$$

Then, $\|L\|_{L^2 \rightarrow L^2} \leq 1$.

Proof. See the section 2.4.1, page 284 of [19]. \square

Lemma 3.4. Let $\frac{\sqrt{2}}{\sqrt{\beta}} \leq T \leq 1$, $\beta \geq 2$, $\alpha \geq 1$ and $|a| \leq \alpha$. Then

$$\|\Psi_T(\cdot) I_a(\cdot)\|_{H_t^{1/2}} \leq C \alpha (\beta + \alpha^2) e^{2\alpha} T^{1/2} \|f\|_{H^{-1/2}}, \quad (3.12)$$

where $I_a(t) := \int_0^t e^{a|t-t'|} f(t') dt'$.

Proof. Rewrite $I_a(t)$ as in the proof of the Lemma 2.4 in [6]. By Fourier inverse transform, we have $I_a(t) = \int_{\mathbb{R}} \widehat{f}(\tau) \frac{e^{i\tau t} - e^{a|t|}}{i\tau - \operatorname{sgn}(t)a} d\tau$. Since, $\frac{1}{\operatorname{sgn}(t)a - i\tau} = \operatorname{sgn}(t)p_a(\tau) + iq_a(\tau)$ where $p_a(\tau) = \frac{a}{a^2 + \tau^2}$ and $q_a(\tau) = \frac{\tau}{a^2 + \tau^2}$ then, replacing τ by t' , we obtain

$$I_a(t) = \operatorname{sgn}(t) \int_{\mathbb{R}} p_a(t') [e^{a|t|} - e^{it't}] \widehat{f}(t') dt' + i \int_{\mathbb{R}} q_a(t') [e^{a|t|} - e^{it't}] \widehat{f}(t') dt' := I_{a,1}(t) + I_{a,2}(t). \quad (3.13)$$

Estimate for $I_{a,1}$. We write

$$\begin{aligned} I_{a,1}(t) &= \operatorname{sgn}(t) \int_{|t'| > 1/T} p_a(t') [e^{a|t|} - e^{it't}] \widehat{f}(t') dt' + \operatorname{sgn}(t) \int_{|t'| \leq 1/T} p_a(t') [e^{a|t|} - e^{it't}] \widehat{f}(t') dt' \\ &:= I_{a,1}^>(t) + I_{a,1}^{\leq}(t) \end{aligned} \quad (3.14)$$

Case 1: $|t'| > 1/T$. In this case $|t'| \asymp \langle t' \rangle$.

$$\Psi_T(t) I_{a,1}^>(t) = \Psi_T(t) \operatorname{sgn}(t) \int_{|t'| > 1/T} p_a(t') [e^{a|t|} - e^{it't}] \widehat{f}(t') dt' = ah_T(t), \quad (3.15)$$

where $h_T(t) = h(t/T)$ and

$$h(t) = \Psi(t) \operatorname{sgn}(t) \int_{|t'| > 1/T} \frac{\widehat{f}(t')}{a^2 + (t')^2} [e^{aT|t|} - e^{iTt't}] dt', \quad (3.16)$$

$$\widehat{h}(t)(\tau) = \int_{|t'| > 1/T} \frac{\widehat{f}(t')}{a^2 + (t')^2} K(a, T, \tau, t') dt' \quad (3.17)$$

with

$$K(a, T, \tau, t') = \int_{\mathbb{R}} \operatorname{sgn}(t) \Psi(t) [e^{aT|t|} - e^{iTt't}] e^{-it\tau} dt. \quad (3.18)$$

Integrating by parts, we have

$$\begin{aligned} K(a, T, \tau, t') &= \int_{\mathbb{R}} \operatorname{sgn}(t) \Psi(t) e^{aT|t|} e^{-it\tau} dt - \int_{\mathbb{R}} \operatorname{sgn}(t) \Psi(t) e^{iTt't} e^{-it\tau} dt \\ &= -\frac{2i}{\tau} - \frac{i}{\tau} \int_{\mathbb{R}} \operatorname{sgn}(t) \left(\frac{d}{dt} \Psi(t) + aT \operatorname{sgn}(t) \Psi(t) \right) e^{aT|t|} e^{-it\tau} dt + \\ &\quad + \frac{2i}{\tau} + \frac{i}{\tau} \int_{\mathbb{R}} \operatorname{sgn}(t) \left(\frac{d}{dt} \Psi(t) + iTt' \Psi(t) \right) e^{i(Tt'-\tau)t} dt \\ &= K_1(a, T, \tau) + K_2(T, \tau, t'), \end{aligned} \quad (3.19)$$

where

$$\begin{aligned}
K_1(a, T, \tau) &= -\frac{i}{\tau} \int_{\mathbb{R}} \operatorname{sgn}(t) \left(\frac{d}{dt} \Psi(t) + aT \operatorname{sgn}(t) \Psi(t) \right) e^{aT|t|} e^{-it\tau} dt \\
&= -\frac{1}{\tau^2} \int_{\mathbb{R}} \operatorname{sgn}(t) \left(\frac{d^2}{dt^2} \Psi(t) + 2aT \operatorname{sgn}(t) \frac{d}{dt} \Psi(t) + (aT)^2 \Psi(t) \right) e^{aT|t|} e^{-it\tau} dt, \quad (3.20) \\
|K_1(a, T, \tau)| &\leq \frac{1}{|\tau|^2} \int_{-2}^2 \left(\left| \frac{d^2}{dt^2} \Psi(t) \right| + 2|a|T \left| \frac{d}{dt} \Psi(t) \right| + (|a|T)^2 |\Psi(t)| \right) e^{aT|t|} dt \leq \frac{C(1+\alpha T)^2 e^{2a}}{|\tau|^2}, \quad (3.21)
\end{aligned}$$

and

$$K_2(T, \tau, t') = \frac{i}{\tau} \int_{\mathbb{R}} \operatorname{sgn}(t) \left(\frac{d}{dt} \Psi(t) + iTt' \Psi(t) \right) e^{i(Tt' - \tau)t} dt \quad (3.22)$$

$$= \frac{2iTt'}{\tau^2} - \frac{1}{\tau^2} \int_{\mathbb{R}} \operatorname{sgn}(t) \left(\frac{d^2 \Psi(t)}{dt^2} + 2iTt' \frac{d\Psi(t)}{dt} + (iTt')^2 \Psi(t) \right) e^{i(Tt' - \tau)t} dt \quad (3.23)$$

$$\begin{aligned}
&= \frac{1}{\tau(Tt' - \tau)} \int_{\mathbb{R}} \operatorname{sgn}(t) \frac{d^2 \Psi(t)}{dt^2} e^{it(Tt' - \tau)} dt + \\
&\quad - \frac{iTt'}{\tau(Tt' - \tau)} \int_{\mathbb{R}} \operatorname{sgn}(t) \frac{d\Psi(t)}{dt} e^{it(Tt' - \tau)} dt - \frac{2iTt'}{\tau(Tt' - \tau)}. \quad (3.24)
\end{aligned}$$

Thus, from (3.22):

$$|K_2(T, \tau, t')| \leq C \frac{|t'|}{|\tau|}, \quad (3.25)$$

from (3.23):

$$|K_2(T, \tau, t')| \leq C \frac{|t'|^2}{|\tau|^2}, \quad (3.26)$$

and from (3.24):

$$\begin{aligned}
|K_2(T, \tau, t')| &\leq \frac{1}{|\tau(Tt' - \tau)|} \int_{-2}^2 \left| \frac{d^2 \Psi(t)}{dt^2} \right| dt + \frac{T|t'|}{|\tau(Tt' - \tau)|} \int_{-2}^2 \left| \frac{d\Psi(t)}{dt} \right| dt + \frac{2T|t'|}{|\tau(Tt' - \tau)|} \\
&\leq \frac{CT|t'|}{|\tau(Tt' - \tau)|}. \quad (3.27)
\end{aligned}$$

In the inequalities above $C = C_\Psi = \max\{\|\Psi\|_{L^\infty}, \|\frac{d}{dt}\Psi\|_{L^\infty}, \|\frac{d^2}{dt^2}\Psi\|_{L^\infty}\}$. From (3.17), (3.19), (3.21) and considering $|\tau| > 1/2$, which implies $\langle \tau \rangle \asymp |\tau|$, we have

$$\begin{aligned}
|\widehat{h}(t)(\tau)| &\leq \int_{|t'| > 1/T} \frac{|\widehat{f}(t')|}{a^2 + (t')^2} |K_1(a, T, \tau)| dt' + \int_{|t'| > 1/T} \frac{|\widehat{f}(t')|}{a^2 + (t')^2} |K_2(T, \tau, t')| dt' \\
&\leq \int_{|t'| > 1/T} \frac{|\widehat{f}(t')|}{[a^2 + (t')^2]} C \frac{(1+\alpha T)^2 e^{2a}}{|\tau|^2} dt' + \int_{1/T < |t'| \leq \beta|\tau|T} \frac{|\widehat{f}(t')|}{a^2 + (t')^2} |K_2(T, \tau, t')| dt' + \\
&\quad + \int_{|t'| \geq \beta|\tau|T} \frac{|\widehat{f}(t')|}{a^2 + (t')^2} |K_2(T, \tau, t')| dt' \\
&= J_1 + J_2 + J_3. \quad (3.28)
\end{aligned}$$

We obtained J_2 and J_3 splitting the set $\{|t'| > 1/T\}$ in $\{1/T < |t'| < \beta|\tau|T\} \neq \{\}$ (because $T \geq \sqrt{2}/\sqrt{\beta}$) and $\{|t'| \geq \beta|\tau|T\}$ where $\beta \geq 2$. We estimate J_1 so,

$$\begin{aligned} J_1 &\leq C \frac{(1 + \alpha T)^2 e^{2a}}{|\tau|^2} \int_{|t'| > 1/T} \frac{|\widehat{f}(t')|}{\langle t' \rangle^{1/2}} \frac{\langle t' \rangle^{1/2}}{(t')^2} dt' \\ &\leq C \frac{(1 + \alpha T)^2 e^{2a}}{|\tau|^2} \|f\|_{H^{-1/2}} \left(\int_{|t'| > 1/T} \frac{1}{(t')^3} dt' \right)^{1/2} \\ &\leq C \frac{(1 + \alpha T)^2 e^{2a} T}{|\tau|^2} \|f\|_{H^{-1/2}}. \end{aligned} \quad (3.29)$$

From (3.27) we obtain

$$|K_2(T, \tau, t')| \leq \frac{CT|t'|}{|\tau(Tt' - \tau)|} \leq \frac{2^{1-\gamma}CT|t'|}{|\tau||\tau|^\gamma |Tt'|^{1-\gamma}} \quad \text{always that } |t'| \geq 2\frac{|\tau|}{T} \quad (3.30)$$

because $|Tt' - \tau| \geq |Tt'| - |\tau| \geq |\tau|$, $|Tt' - \tau| \geq |Tt'| - |\tau| \geq |Tt'|/2$ and so, for $0 \leq \gamma \leq 1$

$$|Tt' - \tau| \geq |\tau|^\gamma \frac{|Tt'|^{1-\gamma}}{2^{1-\gamma}}.$$

Note that $1/T < 2|\tau|/T \leq \beta|\tau|T$. So, to estimate the integral J_3 , since $T \geq \frac{\sqrt{2}}{\sqrt{\beta}}$, then $\frac{2|\tau|}{T} \leq \beta|\tau|T \leq |t'|$, hence

$$\begin{aligned} J_3 &\leq C \int_{|t'| \geq \beta|\tau|T} \frac{|\widehat{f}(t')|}{\langle t' \rangle^{1/2}} \frac{\langle t' \rangle^{1/2} T |t'|}{|\tau|^{1+\gamma} T^{1-\gamma} |t'|^{3-\gamma}} dt' \\ &\leq \frac{CT^\gamma}{|\tau|^{\gamma+1}} \|f\|_{H^{-1/2}} \left(\int_{|t'| > 1/T} \frac{1}{|t'|^{3-2\gamma}} dt' \right)^{1/2} \\ &\leq \frac{CT}{|\tau|^{\gamma+1}} \|f\|_{H^{-1/2}}. \end{aligned} \quad (3.31)$$

To estimate J_2 we are going to use the Schur's lemma 3.3

$$\begin{aligned} J_2 &\leq \int_{1/T < |t'| \leq \beta|\tau|T} \frac{|\widehat{f}(t')|}{(t')^2} |K_2(T, \tau, t')| dt' \\ &\leq \frac{1}{|\tau|^{1/2}} \int_{1/T < |t'| \leq \beta|\tau|T} \frac{|\widehat{f}(t')|}{|t'|^{1/2}} \frac{|\tau|^{1/2} |K_2(T, \tau, t')|}{|t'|^{3/2}} dt', \end{aligned} \quad (3.32)$$

from $\langle \tau \rangle \asymp |\tau|$ and from (3.28) we have

$$\begin{aligned} |\widehat{h(t)}(\tau)| &\leq C \left(\frac{(1 + \alpha T)^2 e^{2a} T}{\langle \tau \rangle^2} + \frac{T}{\langle \tau \rangle^{1+\gamma}} \right) \|f\|_{H^{-1/2}} \\ &\quad + \frac{C}{\langle \tau \rangle^{1/2}} \int_{1/T < |t'| \leq \beta|\tau|T} \frac{|\widehat{f}(t')|}{\langle t' \rangle^{1/2}} \frac{|\tau|^{1/2} |K_2(T, \tau, t')|}{|t'|^{3/2}} dt'. \end{aligned} \quad (3.33)$$

Now, we consider the integral operator $(L_T g)(\tau) = \int_{\mathbb{R}} N_T(\tau, t') g(t') dt'$ where $g(t') = \frac{|\widehat{f}(t')|}{\langle t' \rangle^{1/2}}$ and $N_T(\tau, t') = \frac{|\tau|^{1/2}}{|t'|^{3/2}} |K_2(T, \tau, t')| \chi_{\{\frac{1}{T} < |t'| \leq \beta|\tau|T\}}$. So, we obtain from (3.33) that

$$|\widehat{h(t)}(\tau)| \leq C \left(\frac{(1 + \alpha T)^2 e^{2a} T}{\langle \tau \rangle^2} + \frac{T}{\langle \tau \rangle^{1+\gamma}} \right) \|f\|_{H^{-1/2}} + \frac{C}{\langle \tau \rangle^{1/2}} L_T g(\tau), \quad (3.34)$$

We multiply (3.34) by $\langle \tau \rangle^{1/2}$, take the L^2 norm and obtain

$$\begin{aligned} \left\| \langle \tau \rangle^{1/2} \widehat{h(t)}(\tau) \right\|_{L^2} &\leq C \left(\sqrt{\frac{\beta}{2}} + \alpha \right)^2 T^3 e^{2a} \|f\|_{H^{-1/2}} \left\| \frac{1}{\langle \tau \rangle^{3/2}} \right\|_{L^2} \\ &\quad + C T \|f\|_{H^{-1/2}} \left\| \frac{1}{\langle \tau \rangle^{1/2+\gamma}} \right\|_{L^2} + C \|L_T g(\tau)\|_{L^2}. \end{aligned} \quad (3.35)$$

It is sufficient to prove that the operator L_T is bounded in L^2 . We need to prove that

$$\sup_{\tau} \int_{\mathbb{R}} |N_T(\tau, t')| dt' \leq C(T) \quad \text{and} \quad \sup_{t'} \int_{\mathbb{R}} |N_T(\tau, t')| d\tau \leq C(T)$$

to apply the Schur's Lemma 3.3. We proceed using (3.25)

$$\begin{aligned} \sup_{\tau} \int_{\mathbb{R}} |N_T(\tau, t')| dt' &\leq C \sup_{\tau} \int_{1/T < |t'| \leq \beta|\tau|T} \frac{|\tau|^{1/2}}{|t'|^{3/2}} \frac{|t'|}{|\tau|} dt' \leq C \sup_{\tau} \int_0^{\beta|\tau|T} \frac{|\tau|^{-1/2}}{|t'|^{1/2}} dt' \\ &\leq C \sup_{\tau} |\tau|^{-1/2} T^{1/2} (\beta|\tau|)^{1/2} = C \beta^{1/2} T^{1/2}, \end{aligned} \quad (3.36)$$

and using (3.26)

$$\begin{aligned} \sup_{t'} \int_{\mathbb{R}} |N_T(\tau, t')| d\tau &\leq C \sup_{t'} \int_{|\tau| > 1/2} \frac{|\tau|^{1/2}}{|t'|^{3/2}} \frac{|t'|^2}{|\tau|^2} d\tau \leq C \sup_{t'} \int_{|t'|/\beta T}^{+\infty} \frac{|t'|^{1/2}}{|\tau|^{3/2}} d\tau \\ &\leq C \sup_{t'} \frac{|t'|^{1/2}}{1/2} \left(\frac{|t'|}{\beta T} \right)^{-1/2} = C \beta^{1/2} T^{1/2}. \end{aligned} \quad (3.37)$$

Hence, $\|L_T g\|_{L^2} \leq C \beta^{1/2} T^{1/2}$. Then, from (3.35)

$$\begin{aligned} \left\| \langle \tau \rangle^{1/2} \widehat{h(t)}(\tau) \right\|_{L^2} &\leq C \left[\left(\sqrt{\frac{\beta}{2}} + \alpha \right)^2 T^3 e^{2a} + T \right] \|f\|_{H^{-1/2}} + C \beta^{1/2} T^{1/2} \|g\|_{L^2} \\ \|h\|_{H^{1/2}} &\leq C \left[\left(\sqrt{\frac{\beta}{2}} + \alpha \right)^2 T^3 e^{2a} + T + \beta^{1/2} T^{1/2} \right] \|f\|_{H^{-1/2}} \\ &\leq C (\beta + \alpha^2) e^{2a} T^{1/2} \|f\|_{H^{-1/2}} \end{aligned} \quad (3.38)$$

when $|\tau| > 1/2$. If $|\tau| \leq 1/2$ we use that $|K(a, T, \tau, t')| \leq C(e^{2a} + 1)$ which is consequence of (3.18). So,

$$\begin{aligned} |\widehat{h(t)}(\tau)| &\leq C(e^{2a} + 1) \int_{|t'| > 1/T} \frac{|\widehat{f(t')}|}{a^2 + (t')^2} dt' = C(e^{2a} + 1) \int_{|t'| > 1/T} \frac{|\widehat{f(t')}|}{\langle t' \rangle^{1/2}} \frac{|t'|^{1/2}}{|t'|^2} dt' \\ &\leq C(e^{2a} + 1) \|f\|_{H^{-1/2}} \left(\int_{|t'| > 1/T} \frac{dt'}{|t'|^3} \right)^{1/2} = C(e^{2a} + 1) T \|f\|_{H^{-1/2}}, \end{aligned}$$

and,

$$\int_{|\tau| \leq 1/2} \langle \tau \rangle |\widehat{h(\tau)}|^2 d\tau \leq C(e^{2a} + 1) T \|f\|_{H^{-1/2}} \left(\int_{|\tau| \leq 1/2} \langle \tau \rangle d\tau \right)^{1/2} = C(e^{2a} + 1) T \|f\|_{H^{-1/2}}. \quad (3.39)$$

Thus, adding (3.38) and (3.39), we have that

$$\begin{aligned} \|h\|_{H^{1/2}} &\leq C [(\beta + \alpha^2) e^{2a} T^{1/2} + (e^{2a} + 1) T] \|f\|_{H^{-1/2}} \\ &\leq C [(\beta + \alpha^2) e^{2a} + 1] T^{1/2} \|f\|_{H^{-1/2}}. \end{aligned} \quad (3.40)$$

Finally, from (3.15) and (3.40),

$$\begin{aligned}
\|\Psi_T I_{a,1}^>\|_{H^{1/2}} &= \|ah_T\|_{H^{1/2}} \leq C|a|(T^{1/2}+1)\|h\|_{H^{1/2}} \\
&\leq C\alpha(T^{1/2}+1)[(\beta+\alpha^2)e^{2a}+1]T^{1/2}\|f\|_{H^{-1/2}} \\
&\leq C\alpha[(\beta+\alpha^2)e^{2a}+1]T^{1/2}\|f\|_{H^{-1/2}},
\end{aligned} \tag{3.41}$$

and (3.12) is proved in this case.

Case 2: $|t'| \leq 1/T$. We proceed like in [6], so $\widetilde{\Psi}_T(t) = \text{sgn}(t)\Psi_T(t)$ and

$$\begin{aligned}
(\Psi_T(t)I_{a,1}^<(t))^\wedge(\tau) &= \int_{|t'|\leq 1/T} p_a(t')\widehat{f}(t')\{(\widetilde{\Psi}_T(t)e^{a|t|})^\wedge(\tau) - (\widetilde{\Psi}_T(t)e^{a|t|})^\wedge(\tau-t')\}dt' + \\
&\quad + \int_{|t'|\leq 1/T} p_a(t')\widehat{f}(t')(\widetilde{\Psi}_T(t)[e^{a|t|}-1])^\wedge(\tau-t')dt' \\
&:= I_{a,11}(\tau) + I_{a,12}(\tau).
\end{aligned} \tag{3.42}$$

We can estimate the integral $I_{a,11}$ with the ideas used to prove the Lemma 2.1 in [13].

$$\begin{aligned}
I_{a,11} &= \int_{|t'|\leq 1/T} p_a(t')\widehat{f}(t') \int_{\tau-t'}^\tau \frac{d}{du}(\widetilde{\Psi}_T(t)e^{a|t|})^\wedge(u) du dt' \\
&= \int_{|t'|\leq 1/T} \frac{a t'}{a^2 + (t')^2} \widehat{f}(t') \int_0^1 \frac{d}{d\lambda}(\widetilde{\Psi}_T(t)e^{a|t|})^\wedge(\tau - \lambda t') d\lambda dt'.
\end{aligned} \tag{3.43}$$

We multiply (3.43) by $\langle \tau \rangle^{1/2} \leq C(\langle t' \rangle^{1/2} + |\tau - \lambda t'|^{1/2})$, take the L^2 norm and obtain

$$\begin{aligned}
\|I_{a,11}\|_{H^{1/2}} &\leq C \int_{|t'|\leq 1/T} \frac{|a||t'|\langle t' \rangle^{1/2}|\widehat{f}(t')|}{a^2 + (t')^2} dt' \left\| \frac{d}{dt}(\widetilde{\Psi}_T e^{a|\cdot|})^\wedge \right\|_{L_\tau^2} + \\
&\quad + C \int_{|t'|\leq 1/T} \frac{|a||t'|\widehat{f}(t')|}{a^2 + (t')^2} dt' \left\| |\tau|^{1/2} \frac{d}{dt}(\widetilde{\Psi}_T e^{a|\cdot|})^\wedge \right\|_{L_\tau^2} \\
&\leq C \int_{|t'|\leq 1/T} \frac{|\widehat{f}(t')|}{\langle t' \rangle^{1/2}} \frac{|a||t'|\langle t' \rangle}{a^2 + (t')^2} dt' \left\| (|t|\Psi_T(t)e^{a|t|})^\wedge(\tau) \right\|_{L_\tau^2} + \\
&\quad + C \int_{|t'|\leq 1/T} \frac{|\widehat{f}(t')|}{\langle t' \rangle^{1/2}} \frac{|a||t'|\langle t' \rangle^{1/2}}{a^2 + (t')^2} dt' \left\| |\tau|^{1/2}(|t|\Psi_T(t)e^{a|t|})^\wedge(\tau) \right\|_{L_\tau^2} \\
&\leq C T^{3/2} \|f\|_{H^{-1/2}} \left(\int_{|t'|\leq 1/T} \frac{|a|^2 |t'|^2 \langle t' \rangle^2}{[a^2 + (t')^2]^2} dt' \right)^{1/2} + \\
&\quad + C T \|f\|_{H^{-1/2}} \left(\int_{|t'|\leq 1/T} \frac{|a|^2 |t'|^2 \langle t' \rangle}{[a^2 + (t')^2]^2} dt' \right)^{1/2}
\end{aligned} \tag{3.44}$$

$$\leq C\alpha T \|f\|_{H^{-1/2}} + C\alpha T^{1/2} \|f\|_{H^{-1/2}} \tag{3.45}$$

$$\leq C\alpha T^{1/2} \|f\|_{H^{-1/2}}. \tag{3.46}$$

We obtained (3.44) thanks to Cauchy-Schwartz's inequality and (2.13). (3.45) is consequence from

$$|a|^2 |t'|^4 \leq \alpha^2 |t'|^4 \leq \alpha^2 [a^2 + (t')^2]^2$$

and this implies that the root square of the integrals in (3.44) are bounded by $\sqrt{2}\alpha T^{-1/2}$.

The estimate of the integral $I_{a,12}$ in [6] is not adequate but a small modification is sufficient to obtain a good result. From Case 2 in the proof of the Lemma 2.4 in [6] we know that

$$|(\widetilde{\Psi}_T(t)[e^{a|t|}-1])^\wedge(\tau-t')| \leq C T^2 \frac{|a|}{(1+|\tau|T)^2}. \tag{3.47}$$

So, using Cauchy-Schwartz's inequality

$$\begin{aligned}
|I_{a,12}(\tau)| &\leq C \int_{|t'| \leq 1/T} \frac{|a|^2}{a^2 + (t')^2} |\widehat{f}(t')| \frac{T^2}{(1 + |\tau|T)^2} dt' \\
&= \frac{CT^2}{(1 + |\tau|T)^2} \int_{|t'| \leq 1/T} \frac{|\widehat{f}(t')|}{\langle t' \rangle^{1/2}} \frac{|a|^2 \langle t' \rangle^{1/2}}{a^2 + (t')^2} dt' \\
&\leq \frac{CT^2}{(1 + |\tau|T)^2} \left(\int_{|t'| \leq 1/T} \frac{|a|^4 \langle t' \rangle}{[a^2 + (t')^2]^2} dt' \right)^{1/2} \|f\|_{H^{-1/2}}.
\end{aligned} \tag{3.48}$$

Taking square, multiplying by $\langle \tau \rangle$ and integrating on \mathbb{R} in (3.48) we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \langle \tau \rangle |I_{a,12}(\tau)|^2 d\tau &\leq CT^4 \left\{ \int_{\mathbb{R}} \frac{(1 + |\tau|)}{(1 + |\tau|T)^4} d\tau \right\} \left\{ \int_{|t'| \leq 1/T} \frac{|a|^4 (1 + |t'|)}{[|a| + |t'|]^4} dt' \right\} \|f\|_{H^{-1/2}}^2 \\
&\leq C(\alpha + 1) T \|f\|_{H^{-1/2}}^2
\end{aligned} \tag{3.49}$$

because

$$\int_{\mathbb{R}} \frac{(1 + |\tau|)}{(1 + |\tau|T)^4} d\tau \leq C \left(\frac{1}{T} + \frac{1}{T^2} \right)$$

and

$$\int_{|t'| \leq 1/T} \frac{|a|^4 (1 + |t'|)}{[|a| + |t'|]^4} dt' \leq \frac{1 + |a|}{T} \leq \frac{\alpha + 1}{T}.$$

Hence, from (3.49),

$$\|I_{a,12}\|_{H^{1/2}} \leq C \sqrt{\alpha + 1} T^{1/2} \|f\|_{H^{-1/2}}. \tag{3.50}$$

We conclude from (3.42), (3.46) and (3.50) that

$$\|\Psi_T I_{a,1}^<\|_{H^{1/2}} \leq C(\alpha + \sqrt{\alpha + 1}) T^{1/2} \|f\|_{H^{-1/2}}. \tag{3.51}$$

Note that, from (3.41) and (3.51), we have actually proved that

$$\|\Psi_T I_{a,1}\|_{H^{1/2}} \leq C \{ \alpha [(\beta + \alpha^2) e^{2a} + 1] + \sqrt{\alpha + 1} \} T^{1/2} \|f\|_{H^{-1/2}}, \tag{3.52}$$

which gives (3.12).

Estimate for $I_{a,2}$. The estimate for $I_{a,2}$ is similar to that of $I_{a,1}$, exchanging p_a by q_a and $\widetilde{\Psi}_T$ by Ψ_T . So, we omit its calculation. \square

Corollary 3.1. Let $\frac{\sqrt{2}}{\sqrt{\beta}} \leq T \leq 1$, $\beta \geq 2$, $\alpha \geq 1$ and $-\alpha \leq a \leq 0$.

$$\|\Psi_T I_a\|_{H^{1/2}} \leq C \alpha (\beta + \alpha^2) T^{1/2} \|f\|_{H^{-1/2}}. \tag{3.53}$$

Proof. (3.53) is a direct consequence of (3.52). \square

Lemma 3.5. Let $\frac{\sqrt{2}}{\sqrt{\beta}} \leq T \leq \frac{1}{2}$ and $\beta \geq 8$. Then,

$$\|\Psi_T(\cdot) I_a(\cdot)\|_{H_t^{1/2}} \leq C \alpha (\beta + \alpha^2) T^{1/2} \|f\|_{H^{-1/2}}, \quad \text{if } a < -\alpha, \tag{3.54}$$

and,

$$\|\Psi_T(\cdot) I_a(\cdot)\|_{H_t^{1/2}} \leq C \alpha (\beta + \alpha^2) e^{2\alpha} T^{1/2} \|f\|_{H^{-1/2}}, \quad \text{if } a \leq \alpha. \tag{3.55}$$

Proof. (2.6) and (3.53) give (3.54), and, (3.55) is consequence of (3.54) and (3.12). \square

Proof of Proposition 3.1. From the definition of the $\mathcal{Y}_{s,b}$ norm, we have

$$\begin{aligned}
& \left\| \Psi_T(t) \int_0^t V_\lambda(t-t') F(t') dt' \right\|_{\mathcal{Y}_{s,1/2}}^2 \\
&= \sum_{k \neq 0} \langle k \rangle^{2s} \int_{-\infty}^{\infty} (1 + |\tau|) \left| \left(e^{-itk^3} \Psi_T(t) \int_0^t e^{ik^3(t-t') + \eta|t-t'|\Phi(k)} \widehat{F}(k, t') dt' \right)^\wedge(\tau) \right|^2 d\tau \\
&= \left\| \langle k \rangle^s \left\| \Psi_T(t) \int_0^t e^{\eta|t-t'|\Phi(k)} [e^{-ik^3 t'} \widehat{F}(k, t')] dt' \right\|_{H_t^{1/2}} \right\|_{l_k^2}^2 \\
&\leq \left\| \langle k \rangle^s C \eta \alpha (\beta + (\eta \alpha)^2) e^{2\alpha \eta} T^{1/2} \left\| e^{-ik^3 t} \widehat{F}(k, t) \right\|_{H_t^{-1/2}} \right\|_{l_k^2}^2 \tag{3.56} \\
&= C^2 (\eta \alpha)^2 (\beta + (\eta \alpha)^2)^2 e^{4\eta \alpha} T \left\| \langle k \rangle^s \langle \tau - k^3 \rangle^{-1/2} \widehat{F}(k, \tau) \right\|_{l_k^2 L_\tau^2}^2. \tag{3.57}
\end{aligned}$$

In the inequality (3.56) we apply the Lemma 3.5. (3.57) implies (3.11). \square

4 Local Well-posedness in $H^s(\mathbb{T})$

Consider the λ -periodic initial value problem (1.1) with periodic initial data $u_0 \in H_\lambda^s$, $s \geq -1/2$. We show first that, for arbitrary λ , this problem is well-posed on a time interval of size ~ 1 provided $\|u_0\|_{H_\lambda^{-1/2}}$ is sufficiently small. Then we show by a rescaling argument that (1.1) is locally well-posed for arbitrary initial data $u_0 \in H_\lambda^s$. As mentioned before in Remark 1.1, we restrict our attention to initial data having zero x -mean.

Proof of the Theorem 1.1. Fix $u_0 \in H_\lambda^s$, $s \geq -1/2$ and for $w \in Z^{-1/2}$ define

$$(\mathcal{A}w)(t) = \Psi(t) V_\lambda(t) u_0 - \Psi(t) \int_0^t V_\lambda(t-t') (\Psi(t') w(t')) dt'. \tag{4.1}$$

The bilinear estimate (1.26) shows that $u \in Y^{-1/2}$ implies $\Psi(t) \partial_x(u^2) \in Z^{-1/2}$ so the (nonlinear) operator

$$\Gamma(u) = \mathcal{A}\left(\frac{1}{2} \partial_x(u^2)\right)$$

is defined on $Y^{-1/2}$. Observe that $\Gamma(u) = u$ is equivalent, at least for $t \in [-1, 1]$, to (1.35), which is equivalent to (1.1).

Claim 1. $\Gamma : (\text{bounded subsets of } Y^{-1/2}) \rightarrow (\text{bounded subsets of } Y^{-1/2})$.

Since

$$\Gamma(u) = \Psi(t) V_\lambda(t) u_0 - \Psi(t) \int_0^t V_\lambda(t-t') \left(\frac{\Psi(t')}{2} \partial_x u^2(t') \right) dt',$$

we estimate using (3.2), (3.5) and the bilinear estimate (1.26):

$$\begin{aligned}
\|\Gamma(u)\|_{Y^{-1/2}} &\leq \|\Psi(t) V_\lambda(t) u_0\|_{Y^{-1/2}} + \left\| \Psi(t) \int_0^t V_\lambda(t-t') \left(\frac{\Psi(t')}{2} \partial_x u^2(t') \right) dt' \right\|_{Y^{-1/2}} \\
&\leq C_1 \|u_0\|_{H_\lambda^s} + C_2 \|\Psi(t) \partial_x u^2\|_{Z^{-1/2}} \\
&\leq C_1 \|u_0\|_{H_\lambda^s} + C_2 C_3 \lambda^{0+} \|u\|_{Y^{-1/2}}^2 \tag{4.2}
\end{aligned}$$

and the claim is proved.

Now, we consider the ball

$$\mathfrak{B} = \{w \in Y^{-1/2} : \|w\|_{Y^{-1/2}} \leq C_4 \|u_0\|_{H_\lambda^{-1/2}}\}.$$

Claim 2. Γ is a contraction on \mathfrak{B} if $\|u_0\|_{H_\lambda^{-1/2}}$ is sufficiently small.

We wish to prove that for some $\theta \in (0, 1)$,

$$\|\Gamma(u) - \Gamma(v)\|_{Y^{-1/2}} \leq \theta \|u - v\|_{Y^{-1/2}}$$

for all $u, v \in \mathfrak{B}$. Since $u^2 - v^2 = (u + v)(u - v)$, we can see that

$$\begin{aligned} \|\Gamma(u) - \Gamma(v)\|_{Y^{-1/2}} &\leq \left\| -\Psi(t) \int_0^t V_\lambda(t-t') \frac{\Psi(t')}{2} \partial_x(u^2 - v^2)(t') dt' \right\|_{Y^{-1/2}} \\ &\leq C_2 \|\Psi(t) \partial_x(u + v)(u - v)\|_{Z^{-1/2}} \\ &\leq C_2 C_3 \lambda^{0+} (\|u\|_{Y^{-1/2}} + \|v\|_{Y^{-1/2}}) \|u - v\|_{Y^{-1/2}} \\ &\leq 2 C_2 C_3 C_4 \lambda^{0+} \|u_0\|_{H_\lambda^{-1/2}} \|u - v\|_{Y^{-1/2}}. \end{aligned} \quad (4.3)$$

(4.3) holds because $u, v \in \mathfrak{B}$. Hence, for fixed λ , if we take $\|u_0\|_{H_\lambda^{-1/2}}$ so small such that

$$2 C_2 C_3 C_4 \lambda^{0+} \|u_0\|_{H_\lambda^{-1/2}} \ll 1 \quad (4.4)$$

the contraction estimate is verified.

The preceding discussion establishes well-posedness of (1.1) on a $O(1)$ -sized time interval for any initial data satisfying (4.4). To prove that our result holds for every given data u_0 in H_λ^s and not only for small data as (4.4), let us perform the following scale change

$$v(x, t) = \frac{1}{\sigma^2} u\left(\frac{x}{\sigma}, \frac{t}{\sigma^3}\right) \quad (4.5)$$

where $\sigma \geq \alpha$. So, v is periodic with respect to the x variable with period $\sigma\lambda$, hence for $k \in \mathbb{Z}/\sigma\lambda$

$$\widehat{v}(k) = \frac{1}{\sigma^2} \int_0^{\sigma\lambda} e^{-2\pi i k x} u(x/\sigma) dx = \frac{1}{\sigma} \widehat{u}(k\sigma), \quad (4.6)$$

and v satisfies the equation

$$\sigma^3 v_t(x, t) + \sigma^3 v_{xxx}(x, t) + \sigma^3 v v_x(x, t) + \eta S v(x, t) = 0,$$

where the operator S is defined by $(Sv)^\wedge(k) := -\Phi(\sigma k) \widehat{v}(k)$ and so, $Sv(x, t) = \frac{1}{\sigma^2} Lu\left(\frac{x}{\sigma}, \frac{t}{\sigma^3}\right)$. Hence,

$v_t + v_{xxx} + v v_x + \eta \frac{1}{\sigma^3} S v = 0$. Considering $\widetilde{S} = \frac{1}{\sigma^3} S$, v satisfies

$$\begin{cases} v_t + v_{xxx} + v v_x + \eta \widetilde{S} v = 0 & x \in [0, \sigma\lambda], \quad t \in (0, +\infty) \\ v(x, 0) = v_0(x) = \frac{1}{\sigma^2} u_0\left(\frac{x}{\sigma}\right), \end{cases} \quad (4.7)$$

where

$$(\widetilde{S}v)^\wedge(k) = \frac{1}{\sigma^3} (Sv)^\wedge(k) = -\frac{1}{\sigma^3} \Phi(\sigma k) \widehat{v}(k) \quad \text{and} \quad \frac{1}{\sigma^3} \Phi(\sigma k) \leq \frac{\alpha}{\sigma^3} \leq 1.$$

Finally, consider (1.1) with $\lambda = \lambda_0$ fixed and $u_0 \in H_{\lambda_0}^s$, $s \geq -1/2$. This problem is well-posed on a small time interval $[0, \delta]$ if and only if the σ -rescaled problem (4.7) is well-posed on $[0, \sigma^3 \delta]$. A calculation shows that

$$\|v_0\|_{H_{\sigma\lambda_0}^{-1/2}} \leq \frac{1}{\sigma} \|u_0\|_{H_{\lambda_0}^{-1/2}}.$$

Observe that

$$(\sigma\lambda_0)^{0+} \|v_0\|_{H_{\sigma\lambda_0}^{-1/2}} \leq \frac{(\sigma\lambda_0)^{0+}}{\sigma} \|u_0\|_{H_{\lambda_0}^{-1/2}} \ll 1,$$

provided $\sigma = \sigma(\lambda_0, \|u_0\|_{H_{\lambda_0}^{-1/2}})$ is taken to be sufficiently large. This verifies (4.4) for the problem (4.7) proving well-posedness of (4.7) on the time interval, say $[0, 1]$. Hence, (1.1) is locally well-posed for $t \in [0, \sigma^{-3}]$. \square

References

- [1] B. Alvarez Samaniego, *The Cauchy problem for a nonlocal perturbation of the KdV equation*, Differential Integral Equations, **16** 10 (2003) 1249-1280.
- [2] D. J. Benney, *Long waves on liquids films*, J. Math. Phys., **45** (1996) 150-155.
- [3] H. A. Biagioni, J. L. Bona, R. J. Iorio and M. Scialom, *On the Korteweg-de Vries-Kuramoto-Sivashinsky equation*, Adv. Differential Equations, **1** 1 (1996) 1-20.
- [4] H. A. Biagioni, F. Linares, *On the Benney-Lin and Kawahara Equations*, J. Math. Anal. Appl., **211** (1997) 131-152.
- [5] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolutions equations. II. The KdV-equations*, Geom. Funct. Anal., **3** (1993) 209-262.
- [6] X. Carvajal, M. Panthee, *Well-posedness for some perturbations of the KdV equation with low regularity data*, Electron. J. Differential Equations, **2008** 02 (2008) 1-18.
- [7] X. Carvajal, M. Scialom, *On the well-posedness for the generalized Ostrovsky, Stepanyants and Tsimring equation*, Nonlinear Anal., **62** 2 (2005) 1277-1287.
- [8] W. G. Chen, J. F. Li, *On the low regularity of the Benney-Lin equation*, J. Math. Anal. Appl., **339** (2008) 1134-1147.
- [9] B. I. Cohen, J. A. Krommes, W. M. Tang, M. N. Rosenbluth, *Non-linear saturation of the dissipative trapped-ion mode by mode coupling*, Nuclear Fusion, **69** (1976) 971-992.
- [10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Sharp Global well-posedness for KdV and modified KdV on \mathbb{R} and \mathbb{T}* , J. Amer. Math. Soc., **16** 3 (2003) 705-749.
- [11] S. B. Cui, X. Q. Zhao, *Local well-posedness of the Ostrovsky, Stepanyams and Tsimring equation in Sobolev spaces of negative indices*, Nonlinear Analysis: T. M. A., **70** (2009) 3483-3501.
- [12] S. B. Cui, X. Q. Zhao, *Well-posedness of the Cauchy problem for Ostrovsky, Stepanyams and Tsimring equation with low regularity data*, J. Math. Anal. Appl., **344** (2008) 778-787.
- [13] J. Ginibre, Y. Tsutsumi and G. Velo, *On the Cauchy problem for the Zakharov system*, J. Funct. Anal., **151** 2 (1997) 384-436.
- [14] J. Han and L. Peng, *The well posedness of the dissipative Korteweg-de Vries equations with low regularity data*, Nonlinear Anal., **69** (2008) 171-188.

- [15] C. E. Kenig, G. Ponce and L. Vega, *A bilinear estimate with applications to the KdV equation*, J. Amer. Math. Soc., **9** 2 (1996) 573-603.
- [16] L. Molinet, F. Ribaud, *The Cauchy problem for the dissipative Korteweg de Vries equations in Sobolev spaces of negative order*, Indiana Univ. Math. J., 50 **4** (2001) 1745-1776.
- [17] L. Molinet, F. Ribaud, *On the low regularity of the Korteweg-de Vries-Burgers equation*, Int. Math. Res. Not., **37** (2002) 1979-2005.
- [18] L. A. Ostrovsky, Yu. A. Stepanyants, L. Sh. Tsimring, *Radiation instability in a stratified shear flow*, Int. J. Non-Linear Mech., **19** (1984) 151-161.
- [19] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Ortogonality, and Oscillatory Integrals*, Princeton University Press, (1993).
- [20] T. Tao, *Multilinear weighted convolution of L^2 functions, and applications to nonlinear dispersive equations*, Amer. Jour. of Math., **123** 5 (2001) 839-908.
- [21] J. Topper, T. Kawahara, *Approximate equations for long nonlinear waves on a viscous fluid*, J. Phys. Soc. Japan, **44** (1978) 663-666.
- [22] X. Zhao, *On Low Regularity of the Ostrovsky, Stepanyams and Tsimring Equation*, To apper in J. Math. Anal. Appl., 2011.